HEATING OF MASSIVE BODIES BY VARIABLE-INTENSITY RADIANT HEAT SOURCE

V. V. Salomatov

UDC 536.3

Radiant heating of solid bodies in a variable-temperature medium is investigated. The problem is reduced to a nonlinear Volterra integral equation of the second kind, for which the classical successive-approximation method converges. Subsequent analysis of the resulting functional equation brings to light certain properties of the temperature field; they are used as a basis for an engineering computational method for the given heat-conduction problem.

In connection with problems involving optimal control of heating for massive bodies, much attention is now being given to the development of analytic computational methods for nonstationary temperature fields where the high-intensity heat sources vary in temperature. The problem is one in mathematical physics, with a substantial nonlinearity in the boundary condition; the solution poses serious difficulties. As a rule, when we are concerned with the kinetics of heating, we employ various approximate, basically numerical computational methods. As an example, we have [1, 2], where a computer was used to obtain numerical data on the temperature field in solids heated by radiation, where the source temperature was variable. It was possible to use copious numerical material to analyze the radiant heat flux at the surface of the heated body, and to find a law governing the quasistationary regime in the heating-process dynamics. In [3], segments of cubic parabolas were used to approximate the thermal-conductivity and heat-capacity relationships in order to linearize the system of differential equations describing radiant heating. A similar approach was employed simultaneously and independently in [4]. Hoffmann [5] has given a rather cumbersome solution obtained with a piecewise-linear approximation to the nonlinear intrinsic-emission flux.

The present investigation, which represents a continuation of [1], was undertaken with the aim of obtaining computational equations for the nonstationary temperature field with an arbitrary variation in $\theta_c(\mathrm{Fo})$; analysis of the resulting solutions yields an engineering computational method for the most common types of variation in the temperature of radiant media.

The mathematical model employed for the given physical problem is based on the heat-conduction equation

$$\frac{\partial\theta(X,\ \mathrm{Fo})}{\partial\mathrm{Fo}} = X^{-\nu}\frac{\partial}{\partial X}\left[X^{\nu}\ \frac{\partial\theta(X,\ \mathrm{Fo})}{\partial X}\right] \tag{1}$$

with initial condition

$$\theta(X,\ 0) = \theta_0 \tag{2}$$

and boundary conditions

$$\frac{\partial\theta(0,\ \mathrm{Fo})}{\partial X} = 0, \tag{3}$$

$$\frac{\partial\theta(1,\ \mathrm{Fo})}{\partial X} = \mathrm{Sk}\left[\theta_c^4(\mathrm{Fo}) - \theta^4(1,\ \mathrm{Fo})\right], \tag{4}$$

where $\theta_c(\mathrm{Fo})$ is any differentiable function, $0 \le X \le 1$, $0 \le \mathrm{Fo} \le \infty$.

Approximate Analytic Method. A formal representation of the temperature field can be obtained by reducing the boundary-value problem (1)-(4) to a functional equation. Let us determine the finite integral

S. M. Kirov Polytechnical Institute, Tomsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 2, pp. 292-299, August, 1969. Original article submitted October 10, 1968.

transform that permits us to eliminate the differential operations with respect to X. According to the general theory of finite integral transforms [6-8], the kernel for the direct transformation is represented as

$$\overline{K}(X,\ p)=\frac{1}{c_p}\,\rho(X)\,\overline{K}(X,\ p), \tag{5}$$

where $c_p = \int\limits_0^1 \rho(X)[K(X,p)]^2\,dX$ is the normalizing factor.

The function $\overline{K}(X, p)$ is found from the solution of the corresponding Sturm–Liouville boundary-value problem with the homogeneous boundary conditions

$$\begin{gathered} X^{-\nu}\frac{d}{dX}\left(X^{\nu}\frac{d\overline{K}}{dX}\right)+p^2\overline{K}=0; \\ \left.\frac{d\overline{K}}{dX}\right|_{X=1}=0;\quad \left.\frac{d\overline{K}}{dX}\right|_{X=0}=0,\ \nu=0; \\ \left.\frac{d\overline{K}}{dX}\right|_{X=0}<\infty,\ \nu=1,2. \end{gathered} \tag{6}$$

The eigenfunctions $\overline{K}$ of this problem are orthogonal on the segment [0, 1] with weight $\rho(X)$, while the eigenvalues p are the positive roots of the equation f(p) = 0.

Performing the integral transformation with kernel (5) and interval of integration [0, 1], we transform the initial problem (1)-(4) to an ordinary differential equation with respect to the image, including the specified boundary conditions,

$$\frac{d\overline{\theta}(p,\ \mathrm{Fo})}{d\mathrm{Fo}}=-p^2\overline{\theta}(p,\ \mathrm{Fo})+\Phi\left[p,\ \theta_c^4(\mathrm{Fo}),\theta^4(1,\ \mathrm{Fo})\right], \tag{7}$$

where

$$\begin{gathered} \Phi=(-1)^n\,\mathrm{Sk}\left[\theta_c^4(\mathrm{Fo})-\theta^4(1,\ \mathrm{Fo})\right],\ p=n\pi,\ n=0,\ 1,\ 2,\ \ldots,\quad \nu=0; \\ \Phi=J_0(p)\,\mathrm{Sk}\left[\theta_c^4(\mathrm{Fo})-\theta^4(1,\ \mathrm{Fo})\right],\ J_0'(p)=0,\quad \nu=1; \\ \Phi=\frac{\sin p}{p}\,\mathrm{Sk}\left[\theta_c^4(\mathrm{Fo})-\theta^4(1,\ \mathrm{Fo})\right],\ \mathrm{tg}\,p=p,\quad \nu=2. \end{gathered}$$

The initial condition becomes

$$\overline{\theta}(p,\ \mathrm{Fo})=\frac{\theta_0}{c_p}\int\limits_0^1\rho(X)\,\overline{K}(p,\ X)\,dX. \tag{8}$$

The solution of the transformed problem (7), (8) is

$$\overline{\theta}(p,\ \mathrm{Fo})=\exp[-p^2\mathrm{Fo}]\left\{\overline{\theta}(p,\ 0)+\int\limits_0^{\mathrm{Fo}}\Phi\left[p,\ \theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\exp[p^2\eta]\,d\eta\right\}. \tag{9}$$

Using the inversion formula [6-8]

$$\theta(X,\ \mathrm{Fo})=\sum_{p_i}\overline{\theta}(p_i,\ \mathrm{Fo})\,\overline{K}(p_iX), \tag{10}$$

we obtain an expression for the temperature field

$$\theta(X,\ \mathrm{Fo})=\sum_{p_i}\overline{K}(p_i,\ X)\exp(-p_i^2\mathrm{Fo})\left[\frac{\theta_0}{c_p}\int\limits_0^1\rho(X)\,\overline{K}(p_i,\ X)\,dX+\int\limits_0^{\mathrm{Fo}}\Phi\left[p_i,\ \theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\exp(p_i^2\eta)\,d\eta\right]. \tag{11}$$

Relationship (11) gives a formal representation of the temperature at any point in the body in terms of its value at the surface. Letting X = 1, from (11) we obtain a nonlinear Volterra integral equation of the second kind for $\theta(1, \mathrm{Fo})$:

$$\theta(1,\ \mathrm{Fo}) = \sum_{p_i} \overline{K}(p_i,\ 1)\exp(-p_i^2\mathrm{Fo})\,\frac{\theta_0}{c_p}\int_0^1 \rho(X)\,\overline{K}(p_i,\ X)\,dX$$
$$+\int_0^{\mathrm{Fo}}\sum_{p_i}\overline{K}(p_i,\ 1)\,\Phi\left[p_i,\ \theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\exp\left[-p_i^2(\mathrm{Fo}-\eta)\right]d\eta. \tag{12}$$

The method of successive approximations is one of the most effective ways of solving functional equations of the type (12). Proving the absolute and uniform convergence of the series following the integral sign, we use estimates of the type

$$\sum_{p_i}\overline{K}(p_i,\ 1)\,\Phi\left[p_i,\ \theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\exp\left[-p_i^2(\mathrm{Fo}-\eta)\right]$$
$$\leqslant\left|\sum_{p_i}\overline{K}(p_i,\ 1)\,\Phi\left[p_i,\ \theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\exp\left[-p_i^2(\mathrm{Fo}-\eta)\right]\right|$$
$$<\left|\ \Phi\left[\theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\right|\left|\sum_{p_i}\exp\left[-p_i^2(\mathrm{Fo}-\eta)\right]\right|<\left|\Phi\left[\theta_c^4(\eta),\ \theta^4(1,\ \eta)\right]\right|\left|\frac{\sqrt{\pi}}{2(\mathrm{Fo}-\eta)^{1/2}}\right|, \tag{13}$$

to justify the changed order of summation and integration in (12). It also follows from (13) that the kernel of the integral equation has a singularity at $\mathrm{Fo} = \eta$. As Tikhonov has shown [9] in his investigations of nonlinear Volterra integral equations of the second kind, a singularity of the form $(\mathrm{Fo}-\eta)^{-1/2}$ in the kernel can easily be removed, and the successive-approximation process converges for such equations. Substituting the expressions found for the surface temperature into (11) and performing the uncomplicated integration, we obtain the final solution for the initial problem (1)-(4).

Engineering Method. Further investigation of (11) discloses a certain property of the temperature field. The nonlinear functional equation (11) is difficult to solve owing to the presence of the term of the form $\int_0^{\mathrm{Fo}} \theta^4(1,\ \eta)\exp(p_i^2\eta)\,d\eta$. Let us evaluate this integral,

$$\int_0^{\mathrm{Fo}} \theta^4(1,\ \eta)\exp(p_i^2\eta)\,d\eta \approx \sum_{k=0}^{n-1}\theta^4(1,\ \alpha_k)\exp(p_i^2\alpha_k)\,\Delta\eta_k, \tag{14}$$

where α_k is an arbitrary point on the partial segment $[\eta_k, \eta_{k+1}]$. Writing out the series (14) and isolating its first term with allowance for the initial condition (2), we obtain

$$\theta_0^4\Delta\eta_1 + \sum_{k=1}^{n-1}\theta^4(1,\ \alpha_k)\exp(p_i^2\alpha_k)\,\Delta\eta_k. \tag{15}$$

Remembering that the integrand is strictly increasing, we now have no difficulty in seeing that for moderate values of θ_0, the value of the integral is a very weak function of the temperature level at the beginning of heating. Then the integral (14), which represents the self-radiation of the body, will be evaluated with a certain error. But in the heating of massive bodies, the decisive role is played by internal heat transfer, so that slight distortions in the resultant heat flux owing to approximate determination of the self-radiation may not substantially modify the temperature field.

Thus on the basis of the approximate estimate of (14), using (11), we can establish the existence of a nearly linear relationship of the temperature field with respect to θ_0 for identical values of the controlling criteria Sk, X, Fo, θ_c.

Analysis of temperature fields computed from (11), (12) shows that this property is satisfied up to values $\theta_0 = 0.6$.

The mathematically obtained law is expressed as

$$\frac{\theta(X,\ \mathrm{Fo})-\theta^*(X,\ \mathrm{Fo})}{\theta_0-\theta_0^*}=\frac{\theta^{**}(X,\ \mathrm{Fo})-\theta^*(X,\ \mathrm{Fo})}{\theta_0^{**}-\theta_0^*}, \tag{16}$$

where $\theta^*(X,\ \mathrm{Fo})$, $\theta^{**}(X,\ \mathrm{Fo})$ are the respective dimensionless temperatures for the minimum (θ_0^*) and the maximum (θ_0^{**}) values of initial temperature for the same values of Sk, X, Fo, θ_c. The structure of (16)

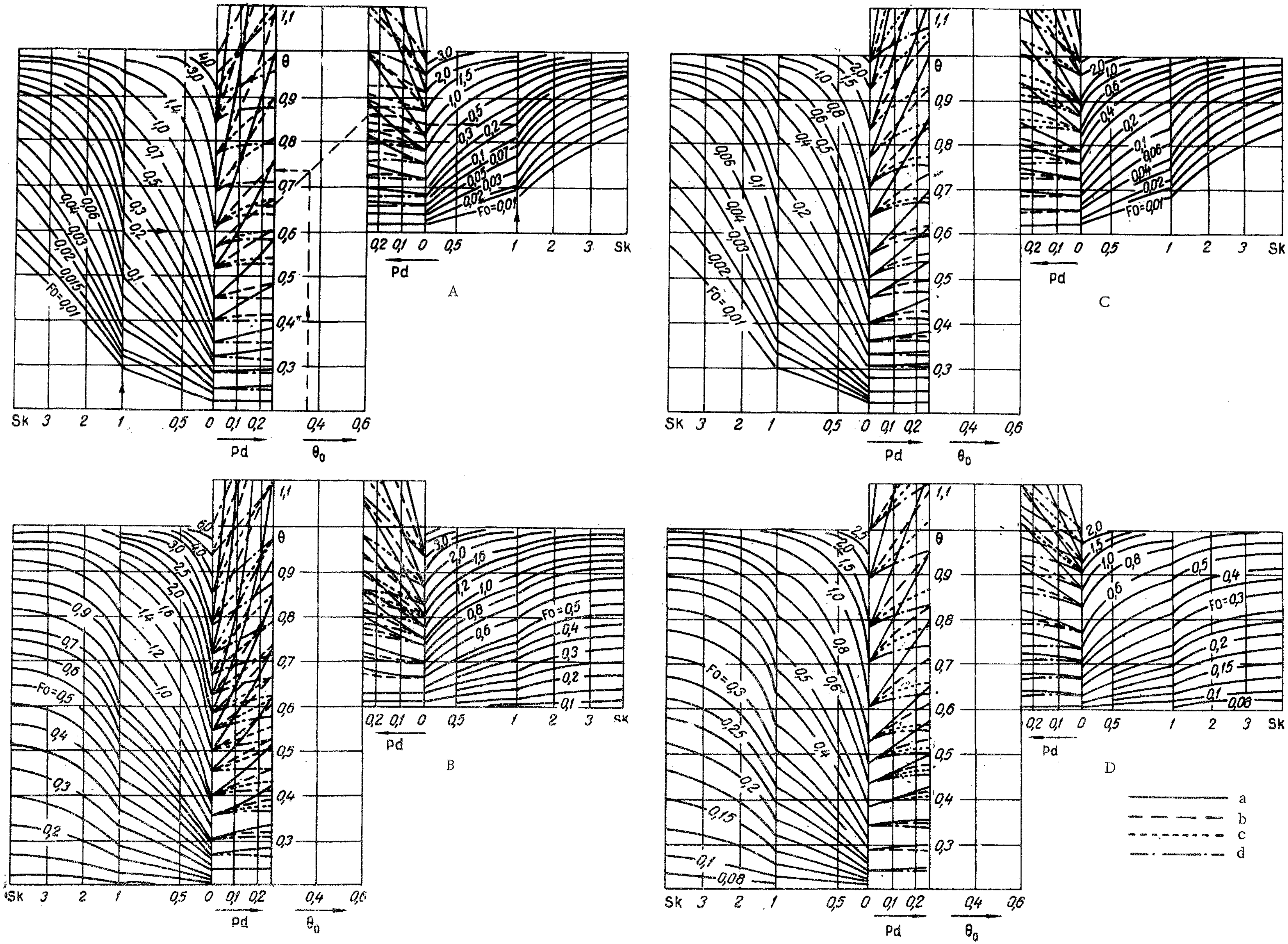

Fig. 1. Nomogram for determining relative temperature at surface of unbounded plate (A), at center of unbounded plate (B), at surface of infinite cylinder (C), and on axis of infinite cylinder (D): a) Sk = 0.25; b) 0.5; c) 1.0; d) 3.0.

is such that when the temperature distributions are known, for two values of the initial temperature we can compute the temperature field for arbitrary θ_0,

$$\theta(X, \mathrm{Fo}) = \theta^*(X, \mathrm{Fo}) + (\theta_0 - \theta_0^*) \frac{\theta^{**}(X, \mathrm{Fo}) - \theta^*(X, \mathrm{Fo})}{\theta_0^{**} - \theta_0^*}. \tag{17}$$

The more effective relationship (17) can be realized graphically; to do this, we must know the solutions for two values of θ_0. We obtained the solutions by numerical intergration of Eqs. (11), (12) on an M-20 computer, with the process parameters Sk = 0-4, Fo = 0-∞, with a linear variation in the emitter temperature, $\theta_c(\mathrm{Fo}) = 1 + \mathrm{PdFo}$, where Pd = 0-0.25; θ_0 was taken equal to 0.2 and 0.6.

Figure 1 (A-D) shows computational nomograms for determining the dimensionless temperatures at the surface and at the center of an unbounded plate, and also at the surface and on the axis of an infinite cylinder.

Comparing our calculated results with reference data obtained by computer, we see that for the relative temperatures, the deviations do not exceed 1-2%, and tend to make the results too low. It should be noted that a similar approach can be used to investigate radiant heat-conduction processes in multidimensional bodies (a cylinder of finite dimensions, a prism, a parallelepiped), and can also be employed for other types of variation in the temperature of the radiating medium (exponential, parabolic, etc.).

We also note that the proposed nomograms can also be used when $\theta_0 < 0.2$. Computational accuracy can be improved if we employ solutions for a narrower range of values of the initial temperature.

The key to the nomograms is shown in Fig. 1 for the following process parameters: Pd = 0.15; Sk = 1.0; Fo = 0.15; $\theta_0 = 0.35$. The desired relative temperature at the surface of an unbounded plate is $\theta_{sur} = 0.736$.

NOTATION

$\theta = T/T_{c0}$	is the relative temperature, i.e., the ratio of the instantaneous temperature to the initial temperature of the heat source;
$\mathrm{Sk} = \sigma_a (T_{c0}/100)^3 R/\lambda \cdot 100$	is the radiation Stark number;
ν	is a coefficient that allows for the shape of the body, and is numerically equal to 0, 1, and 2 for the plane, cylindrical, and spherical problems;
$\mathrm{Pd} = kR^2/T_{c0}\,a$	is the Predvoditelev number for heating at a constant rate.

LITERATURE CITED

1. V. V. Salomatov and G. P. Boikov, Inzh.-Fiz. Zh., 8, No. 3 (1965).
2. V. V. Ivanov, V. V. Salomatov, and V. Yu. Chekhovich, Izv. Akad. Nauk SSSR, Énergetika i Transport, No. 1 (1967).
3. V. V. Salomatov, Izv. Vys. Uch. Zav., Fizika, No. 5 (1965).
4. L. A. Brovkin, Izv. Vys. Uch. Zav., Energetika, Nos. 3-4 (1965).
5. A. Hoffmann, Acta Technica, 40, Nos. 3-4 (1962).
6. N. S. Koshlyakov et al., The Fundamental Differential Equations of Mathematical Physics [in Russian], GIFML (1962)
7. Dzh. K. Tranter, Integral Transforms in Mathematical Physics [in Russian], Gostekhizdat (1956).
8. G. A. Grinberg, Selected Problems in the Mathematical Theory of Electrical and Magnetic Phenomena [in Russian], Izd. AN SSSR (1948).
9. A. N. Tikhonov, Byull. MGU, Ser. A, Mat. i Mekh., No. 8 (1938).